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Exact Solutions for Self–Dual Yang–Mills and Self–Dual Tensor Multiplets on Gravitational Instanton Background [1]

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Abstract

We give exact solutions for a recently developed $N = 1$ locally supersymmetric self-dual gauge theories in $(2+2)$-dimensions. We give the exact solutions for an $SL(2)$ self-dual Yang-Mills multiplet and what we call "self-dual tensor multiplet" on the gravitational instanton background by Eguchi-Hanson. We use a general method to get an $SL(2)$ self-dual Yang-Mills solution from any known self-dual gravity solution. Our result is the first example of exact solutions for the coupled system of these $N = 1$ locally supersymmetric self-dual multiplets in $(2+2)$-dimensions, which is supposed to have strong significance for integrable models in lower-dimensions upon appropriate dimensional reduction or truncation. We also inspect the consistency of our exact solutions as a background for $N = 2$ superstring coupled to the Wess-Zumino-Witten term in σ-model formulation.

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1. *Introduction.* Recently there has been important observation in $N=2$ superstring theory [1] that the massless background fields for $N=2$ superstring are to be the self-dual Yang-Mill (SDYM) [2] or self-dual gravity (SDG) fields. Since the space-time supersymmetry is to be built-in in such superstring theory, it is natural to expect that these background fields have to be also *supersymmetric*. Motivated by this development, we have constructed in our recent papers [3-6] self-dual *supersymmetric* YM (SDSYM) theories and self-dual supergravity (SDSG) theories in four-dimensional space-time with the signature $(+,+,-,-)$.[2] Another strong motivation to study SDYM theory is from the conjecture [7] that *all* exactly soluble (bosonic) models in lower-dimensions can be obtained from the SDYM theory [8]. Moreover the W_∞-algebra [9] is also likely to be connected to the SDG theory in $D=(2,2)$.

According to a more recent analysis [10], open $N=2$ superstring allows only $N=4$ SDSYM, while closed $N=2$ superstring allows only $N=8$ SDSG as consistent target space-time backgrounds, provided that the background is described by a single irreducible superfield. Even if these "maximal" supersymmetries may be singled out by consistency for the irreducibility in the $N=2$ superstring, it will be still important to consider some truncated supersymmetries of these backgrounds, such as $N\le 4$ SDSG [5] or $N\le 2$ SDSYM [3,5], from the viewpoint of soluble systems in lower-dimensions [7].

In our recent paper [6] we gave an exact solution for the gaugino field in the *global* $N=2$ SDSYM theory on the bosonic YM instanton background. We have seen that the gaugino solution is generated by a *globally* supersymmetric transformation of the gaugino on the bosonic instanton background.

In this Letter, we give exact solutions for the $N=1$ SDYM multiplet for the gauge group $SL(2)\approx SO(1,2)$ coupled to what we call "self-dual tensor multiplet" (SDTM) and the $N=1$ SDSG on the space-time background of Eguchi-Hanson instanton metric [11]. Since our method is based on the general feature of the SDG and SDYM system, it will also provide a general algorithm generating exact solutions for $SL(2)$ SDYM, whenever a purely gravitational solution for the SDG is given.

2. *Field Equations.* We first review our relevant field equations in the system. The field content of the $D=(2,2)$, $N=1$ SDSG is $(e_\mu{}^m,\, \widetilde\psi_\mu{}^{\dot\alpha})$, that of the SDSYM is $(A_\mu{}^I,\, \lambda_\alpha{}^I)$ and that of the SDTM is $(B_{\mu\nu},\, \Phi,\, \chi_\alpha)$. The Φ is the usual dilaton, while $B_{\mu\nu}$ couples to the $N=2$ superstring [4] *via* the Wess-Zumino-Witten term. In this paper we use the indices $\mu,\nu,\dots=1,\dots,4$ for the *curved* world coordinates, and $m,n,\dots=1,\dots,4$ for the local Lorentz coordinates. For the spinors we use the same convention as in Refs. [3-6], namely

[2] We denote this space-time by $D=(2,2)$.

$\alpha, \beta, \cdots = 1, 2$ and $\dot\alpha, \dot\beta, \cdots = \dot1, \dot2$ are respectively for the *chiral* and *anti-chiral* components. All the *anti-chiral* spinors are denoted by the *tilde*, such as $\widetilde\psi_\mu{}^{\dot\alpha}$. We use the 2×2 matrices $(\gamma^\mu)_{\alpha\dot\beta}$ and $(\gamma^\mu)_{\dot\alpha\beta}$ instead of $(\sigma^{\underline{m}})_{\alpha\dot\beta}$ and $(\widetilde\sigma^{\underline{m}})_{\dot\alpha\beta}$ in Refs. [3-6]. The indices $I, J, \cdots = 1, 2, 3$ are for the adjoint representations for the $SL(2)$ gauge group.

To get the exact solutions, we use the *canonical* set[3] of field equations [12], because it has *no* torsion making the self-duality (SD) of the Riemann tensor manifest. One important point about the supersymmetric self-dual system in general is that the bosonic field equations in our system stay exactly the same as the *non-supersymmetric* theory, especially for the SDSYM and SDSG fields, while the bosonic fields for the SDTM satisfy the peculiar field equations:

$$i(\gamma^\nu)_{\alpha\dot\beta}\widetilde T_{\mu\nu}{}^{\dot\beta} = 0 \quad , \tag{2.1}$$

$$i(\gamma^\mu)^\beta{}_{\dot\alpha} D_\mu \chi_\beta + \frac{i}{2\sqrt3}(\gamma^\mu\chi)_{\dot\alpha}\partial_\mu\Phi - \frac{1}{2\sqrt3}(\gamma^{\mu\nu}\widetilde\lambda^I)_{\dot\alpha} e^{-\Phi/\sqrt3} F_{\mu\nu}{}^I = 0 \quad , \tag{2.2}$$

$$i(\gamma^\mu)_\alpha{}^{\dot\beta} D_\mu \widetilde\lambda_{\dot\beta}{}^I - \frac{i}{2\sqrt3}(\gamma^\mu\widetilde\lambda^I)_\alpha \partial_\mu\Phi = 0 \quad , \tag{2.3}$$

$$R^{\mu\nu\rho\sigma} = \frac12 e^{-1}\epsilon^{\mu\nu\tau\omega} R_{\tau\omega}{}^{\rho\sigma} \quad , \qquad R_{\mu\nu} = 0 \quad , \tag{2.4}$$

$$\Box\,\Phi + \frac{2}{\sqrt3}(\partial_\mu\Phi)^2 - \frac{1}{2\sqrt3}e^{-2\Phi/\sqrt3}F_{\mu\nu}{}^I F^{\mu\nu}{}_I = 0 \quad , \qquad G^{\mu\nu\rho} = e^{2\Phi/\sqrt3}\, e^{-1}\epsilon^{\mu\nu\rho\sigma}\,\partial_\sigma\Phi \quad , \tag{2.5}$$

$$F^{\mu\nu\, I} = \frac12 e^{-1}\epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}{}^I \quad , \qquad D_\mu(eF^{\mu\nu\, I}) = 0 \quad . \tag{2.6}$$

The derivative D_μ is both Lorentz and gauge-covariant, and $\widetilde T_{\mu\nu}{}^{\dot\beta}$ is the gravitino field strength. The field strength $G_{\mu\nu\rho}$ of $B_{\mu\nu}$ contains the Chern-Simons (CS) term:

$$G_{\mu\nu\rho} \equiv 3\partial_{[\mu}B_{\nu\rho]} - \sqrt3\left[F_{[\mu\nu}{}^I A_{\rho]\, I} - \frac13 f_{IJ}{}^K A_\mu{}^I A_\nu{}^J A_{\rho K}\right] \quad . \tag{2.7}$$

Here the antisymmetrizations with the symbols $[\;]$ are always *normalized*. The special factor with $\sqrt3$ comes from the normalization in our *canonical* set of field equations [12]. The appearance of the special factors like $e^{2\Phi/\sqrt3}$ is also due to our canonical system. The second equation in (2.5) implies a "generalized" SD, namely the third-rank tensor $G_{\mu\nu\rho}$ is *dual* to the first-rank field strength $\partial_\mu\Phi$. In the above field equations all the field strengths need *no supercovariantization* due to the special property of the self-dual supersymmetry. For example, the field strength $F_{\mu\nu}{}^I$ has *no* gravitino-dependent term, because the gravitino $\widetilde\psi_\mu{}^{\dot\alpha}$ lacks its *chiral* partner $\psi_\mu{}^\alpha$, so that there is no possibility such as $\psi\gamma\widetilde\lambda$-terms. Since $SL(2)$ is a non-compact group, we always need its metric $g_{IJ} = \mathrm{diag.}\,(1,-1,-1)$ for the contractions of the indices $I, J, \cdots = 1, 2, 3$.

[3]The word *canonical* comes from the fact that the *kinetic* terms in the original *non-self-dual* lagrangian before imposing the SD conditions have the standard coefficients. The difference between *canonical* and non-canonical versions is just a matter of field-redefinitions, but in practice the former set has such an advantage as the vanishing torsion, *etc.* For other details, see Ref. [12].

3. *Exact Solutions.* We now give the exact solutions for our field equations (2.1) - (2.6). We start with the field equations (2.1) and (2.4) for the SDSG. Eq. (2.4) is satisfied by what is called Eguchi-Hanson (EH) gravitational instanton solution [11], modified for our $D = (2,2)$. Such EH metric is given by

$$\begin{aligned}
&g_{11} = \frac{1}{1-\frac{1}{r^4}} \quad , \quad g_{22} = \frac{r^2}{4}\left(1 - \frac{1}{r^4}\cosh^2\vartheta\right) \quad , \quad g_{24} = \frac{1}{r^4}\left(1-\frac{1}{r^4}\right)\cosh\vartheta \quad , \\
&g_{33} = -\frac{1}{4}r^2 \quad , \quad g_{44} = \frac{r^2}{4}\left(1-\frac{1}{r^4}\right) \quad ,
\end{aligned} \tag{3.1}$$

where our choice of coordinates is $(x^\mu) = (r,\, \varphi,\, \vartheta,\, \psi)$, and $0 \le \varphi < 2\pi,\ 0 \le \vartheta < \infty,\ 0 \le \psi < 2\pi$.

The gravitino equation (2.1) is satisfied by the trivial solution $\widetilde\psi_\mu{}^{\dot\alpha} = 0$, and this does *not* pose any problem for the following reason. First, we can easily show that the *pure-gauge* solution

$$\widetilde\psi_\mu{}^{\dot\alpha} = D_\mu \widetilde\lambda^{\dot\alpha} \quad , \tag{3.2}$$

with an arbitrary space-time dependent spinor $\widetilde\lambda$ together with the EH background (3.1) satisfies our field equations (2.1), due to the identity $R_{[\mu\nu\rho]}{}^\sigma \equiv 0$. Now recall that the supertranslation rule

$$\delta\widetilde\psi_\mu{}^{\dot\alpha} = D_\mu \widetilde\epsilon^{\dot\alpha} \quad , \quad \delta e_\mu{}^m = -i(\epsilon\gamma^m\widetilde\psi_\mu) \quad , \tag{3.3}$$

which can completely gauge away the above solution (3.2). This means that by choosing an appropriate frame of supersymmetry, we can put the background of the gravitino $\widetilde\psi_\mu$ to be zero.[4] Therefore we simply set the gravitino to be zero from now on.

Our next task is to solve the SDSYM field equations (2.3) and (2.6). We can easily find a non-trivial solution to (2.6) on our EH background (3.1). The strategy is to utilize the fact that we can choose a gauge group $SL(2)$, which coincides with one of the subgroups of the Lorentz group in the SDG: $SO(2,2) \approx SL(2) \otimes SL(2)$. This method has been known for the Euclidean case for getting a SDYM solution for the gauge group $SU(2)$ out of any known SDG solution in the Euclidean space-time [13].

To be more specific, we can identify the Lorentz connection $\omega_\mu{}^{mn}$ for the EH instanton background (3.1) with our YM gauge field $A_\mu{}^I$ as

$$\begin{aligned}
\omega_\mu{}^{(1)(2)} = \omega_\mu{}^{(3)(4)} &\rightarrow A_\mu{}^1 \quad , \\
\omega_\mu{}^{(1)(3)} = \omega_\mu{}^{(2)(4)} &\rightarrow A_\mu{}^2 \quad , \\
\omega_\mu{}^{(1)(4)} = \omega_\mu{}^{(3)(2)} &\rightarrow A_\mu{}^3 \quad .
\end{aligned} \tag{3.4}$$

[4]Of course, however, this does *not* exclude the existence of other gauge-non-trivial solutions for the gravitino. Our choice is just one choice of gauge-trivial family of exact solutions for the gravitino. Other non-trivial gravitino solutions are yet to be studied in the future.

Here we use the indices in the parentheses such as $(1), \cdots, (4)$ for the *flat* Lorentz indices $m, n, \cdots$, distinguished from the *curved* ones $\mu, \nu, \cdots = 1, \cdots, 4$ without parentheses. Notice that this identification has been made possible, owing to the manifest SD for the mn-indices of ${\omega_\mu}^{mn}$.[5] The $SL(2)$ gauge roup has the generators T_I satisfying

$$[T_1, T_2] = -2T_3 \quad , \qquad [T_2, T_3] = +2T_1 \quad , \qquad [T_3, T_1] = -2T_2 \quad . \tag{3.5}$$

Relevantly, we can rewrite them in terms of the familiar Virasoro algebra notation:

$$[L_0, L_{\pm 1}] = \mp L_{\pm 1} \quad , \qquad [L_1, L_{-1}] = 2L_0 \quad , \tag{3.6}$$

through the identification

$$T_1 = -2L_0 \quad , \qquad T_2 = L_1 + L_{-1} \quad , \qquad T_3 = -L_1 + L_{-1} \quad . \tag{3.7}$$

For our purpose of utilizing EH instanton (3.1), the T_I 's in (3.5) is more advantageous.

Performing the identifications (3.4) we get the solution for ${A_\mu}^I$:

$$\begin{aligned}
&{A_2}^2 = -\tfrac{1}{2}\sqrt{1-\frac{1}{r^4}}\sinh\vartheta\cos\psi \quad , \qquad {A_2}^3 = -\tfrac{1}{2}\sqrt{1-\frac{1}{r^4}}\sinh\vartheta\sin\psi \quad , \\
&{A_2}^1 = \tfrac{1}{2}\left(1+\frac{1}{r^4}\right)\cosh\vartheta \quad , \\
&{A_3}^2 = \tfrac{1}{2}\sqrt{1-\frac{1}{r^4}}\sin\psi \quad , \qquad {A_3}^3 = -\tfrac{1}{2}\sqrt{1-\frac{1}{r^4}}\cos\psi \quad , \\
&{A_4}^1 = \tfrac{1}{2}\left(1+\frac{1}{r^4}\right) \quad ,
\end{aligned} \tag{3.8}$$

and all other components are zero. The satisfaction of the SD condition (2.6) is easily confirmed for the field strength

$${F_{\mu\nu}}^I = \partial_\mu {A_\nu}^I - \partial_\nu {A_\mu}^I + {f_{JK}}^I {A_\mu}^J {A_\nu}^K \quad . \tag{3.9}$$

A remarkable point is that despite of the EH gravitational instanton background, the SD condition (2.6) *does* hold for our SDYM instanton. For explicitness, we give the YM field

[5]Tish is *not* generally true for any Lorentz connection for a given self-dual Riemann tensor [14].

strength:

$$
\begin{aligned}
&F_{12}{}^1 = -\frac{2}{r^5}\cosh\vartheta \ , \quad F_{14}{}^1 = -\frac{2}{r^5} \ , \quad F_{23}{}^1 = -\frac{1}{r^4}\sinh\vartheta \ , \\
&F_{12}{}^2 = -\frac{\sinh\vartheta\cos\psi}{r^5\sqrt{1-\frac{1}{r^4}}} \ , \quad F_{23}{}^2 = -\frac{\sqrt{1-\frac{1}{r^4}}\cosh\vartheta\cos\psi}{2r^4} \ , \\
&F_{34}{}^2 = \frac{\sqrt{1-\frac{1}{r^4}}\cos\psi}{2r^4} \ , \quad F_{24}{}^2 = \frac{\sqrt{1-\frac{1}{r^4}}\sinh\vartheta\sin\psi}{2r^4} \ , \\
&F_{12}{}^3 = -\frac{\sinh\vartheta\sin\psi}{r^5\sqrt{1-\frac{1}{r^4}}} \ , \quad F_{13}{}^3 = -\frac{\cos\psi}{r^5\sqrt{1-\frac{1}{r^4}}} \ , \\
&F_{23}{}^3 = -\frac{1}{2r^4}\sqrt{1-\frac{1}{r^4}}\cosh\vartheta\sin\psi \ , \quad F_{24}{}^3 = -\frac{1}{2r^4}\sqrt{1-\frac{1}{r^4}}\sinh\vartheta\cos\psi \ , \\
&F_{34}{}^3 = \frac{\sqrt{1-\frac{1}{r^4}}\sin\psi}{2r^4} \ ,
\end{aligned}
\tag{3.10}
$$

and all other independent components are zero.

The important ingredient about our prescription above is the special role played by the $SL(2)$ indices ${\scriptstyle mn}$ in $\omega_\mu{}^{mn}$ or $R_{\mu\nu}{}^{mn}$, as if they were "internal" $SL(2)$ gauge symmetry. This is the $SO(2,2)$ analog of the usual Euclidean case $SO(4) \approx SU(2) \otimes SU(2)$ [13].

As for the gaugino equation (2.3), we simply put the gaugino to zero, similarly to the gravitino case. This can be done consistently with supersymmetry, as long as such an ansatz satisfies all the field equations. This is also reasonable from the fact that we have already fixed the freedom of supersymmetry, when the gravitino is put to zero, and eventually the background has *no* manifest supersymmetry.

Our remaining field equations are for the SDTM. As in the case of the gaugino, we can satisfy (2.2) by the trivial solution $\chi_\alpha = 0$, like the gravitino and gaugino. Eq. (2.5) is rather easily solved now. First we rewrite (2.5) in the form

$$
\Box\,\phi - \tfrac{1}{3} F_{\mu\nu}{}^I F^{\mu\nu}{}_I = 0 \ , \tag{3.11}
$$

where

$$
\phi \equiv \exp\left(\tfrac{2}{\sqrt{3}}\Phi\right) \ . \tag{3.12}
$$

Inserting the solution (3.10) into (3.11), we get

$$
\phi''(r) + \frac{3r^4+1}{r(r^4-1)}\phi'(r) = \frac{32}{r^8(r^4-1)} \ , \tag{3.13}
$$

where we have assumed that ϕ depends *only* on r, and each prime is for the derivative d/dr. This can be easily solved by

$$
\phi = -\frac{2(3r^4+1)}{3r^6} + \frac{a-4}{4}\ln\left(\frac{r^2-1}{r^2+1}\right) + 1 \ , \tag{3.14}
$$

where we imposed the boundary condition $\Phi(r \to \infty) = 0$, and a is an arbitrary constant corresponding to the general solution of (3.13).

The $B_{\mu\nu}$ field is now solved as (3.18) below by the help of (2.5) as follows. The non-vanishing independent component of $G_{\mu\nu\rho}$ obtained from (2.5) with (3.14) is

$$G_{234} = \frac{\sqrt{3}(4 - ar^8)}{16r^8} \sinh\vartheta \quad . \tag{3.15}$$

Here the CS term played a peculiar role for the integrability for the potential field $B_{\mu\nu}$, as guaranteed by the Bianchi identity:

$$\partial_{[\mu} G_{\nu\rho\sigma]} = -\frac{\sqrt{3}}{2} F_{[\mu\nu}{}^{I} F_{\rho\sigma]\,I} \quad . \tag{3.16}$$

valid for our solution (3.10).

To summarize our results, we have the solutions

$$\Phi = \frac{\sqrt{3}}{2} \ln\left|1 - \frac{2(3r^4+1)}{3r^6} + \frac{a-4}{4}\ln\left(\frac{r^2-1}{r^2+1}\right)\right| \quad , \tag{3.17}$$

$$B_{24} = \frac{4+3a}{16\sqrt{3}} \cosh\vartheta + b\,(\varphi,\,\psi) \quad , \tag{3.18}$$

for the SDTM, and the solution (3.10) or (3.8) for the SDYM on the EH background (3.1). The $b\,(\varphi,\,\psi)$ is an arbitrary function only of φ and ψ.

Due to the non-compactness of our space-time, the various topological integrals [14] for our solutions do not converge. This is mainly caused by the integral $\int_0^\infty d\vartheta\, \sinh\vartheta \to \infty$. The action or hamiltonian (before imposing the SD conditions [12], or in what we call Parkes-Siegel (PS) formulation [6,10]) for our exact solutions is also divergent due to a boundary term with the same ϑ-integral. However, we expect that an appropriate Wick-rotation can lead our space to the compact Euclidian space-time, which replaces the above integral by the *finite* one: $\int_0^\pi d\theta\, \sin\theta = 2$. In fact, we get the Euler number [14] after such a replacement:

$$\begin{aligned} \chi(M) &= \frac{1}{16\pi^2}\int_{\mathrm{M}} d^4x\, R^{mn} \wedge R_{mn} - \frac{1}{16\pi^2}\int_{\partial\mathrm{M}} \epsilon^{mnrs}\left[\omega_{mn} \wedge R_{rs} - \tfrac{2}{3}\omega_{mn} \wedge \omega_r{}^t \wedge \omega_{ts}\right] \\ &= -\tfrac{1}{2}(-3) - (-\tfrac{1}{2}) = +2 \quad . \end{aligned} \tag{3.19}$$

From these viewpoints, we regard our exact solutions as equally important as the Euclidian cases [14].

Another interesting feature of our solutions is the topological significance of the $B_{\mu\nu}$-field related to the instanton number *via* the CS term. In particular, we have

$$\begin{aligned} \int_{\mathrm{M}} F^I \wedge F_I &= -\sqrt{3}\int_{\mathrm{M}} dG = -\sqrt{3}\int_{\partial\mathrm{M}} G \\ &= \frac{3}{2}\int_{\partial\mathrm{M}} *d\left(e^{2\Phi/\sqrt{3}}\right) = \frac{3}{2}\int_{\mathrm{M}} d^4x\, e\Box\,\phi \quad . \end{aligned} \tag{3.20}$$

The penultimate equality is due to the SD in the SDTM.

The next natural question is the compatibility and relationship of our exact solution with $N=2$ string theory. In our previous paper [4], we have given a possible Green-Schwarz (GS) σ-model formulation. In our present case we have to rearrange the action, such that the coupling is consistent with our *canonical* fields. Leaving all the details to Ref. [12], we give the Wess-Zumino-Witten (WZW) term in the GS σ-model relevant to our discussion here:

$$I_{\rm WZW} = \int d^2\sigma \left[-\frac{1}{2\pi}\frac{1}{\sqrt{3}}\,\epsilon^{ij}\Pi_i{}^A\Pi_j{}^B B_{BA}\right] \quad , \tag{3.21}$$

where $\Pi_i{}^A \equiv (\partial_i Z^M)E_M{}^A$ with the $D=(2,2)$ (inverse)vielbein $E_M{}^A$, and the superspace coordinates Z^M. The indices ${\scriptstyle i,\ j,\ \cdots\ =\ 1,\ 2}$ are for the world-sheet curved coordinates. In the absence of fermionic backgrounds we can replace $\Pi_i{}^A$ by $(\partial_i X^\mu)e_\mu{}^m$ (Neveu-Ramond-Schwarz σ-model). The special factor $1/\sqrt{3}$ is for the normalization fixed by our *canonical* fields [12]. To see the effect of $B_{\mu\nu}$ on the WZW-term, we regard the r-coordinate as a "time" variable for our "instanton" solution (3.10). Considering also the appropriate normalization factors, the effect of such instanton at time r yields the exponent[6]

$$\begin{aligned} P(r) &= \frac{1}{2\pi}\frac{1}{\sqrt{3}}\int d^2\sigma\,\epsilon^{ij}(\partial_i X^\mu)(\partial_j X^\nu)B_{\mu\nu} \\ &= \frac{1}{2\pi}\frac{1}{\sqrt{3}}\int d^2\sigma\int_0^1 du\,\epsilon^{\hat i\hat j\hat k}(\partial_{\hat i}\widehat X^\mu)(\partial_{\hat j}\widehat X^\nu)(\partial_{\hat k}\widehat X^\rho)\widehat G_{\mu\nu\rho} \\ &= \frac{3(4-ar^8)}{16\pi r^8}\int_0^{2\pi}d\varphi\int_0^{2\pi}d\psi\int_0^\pi d\theta\sin\theta \\ &= \frac{6\pi}{r^8}-\frac{3a\pi}{2} \quad , \end{aligned} \tag{3.22}$$

in the string path-integral [16]. We have used what is called "Vainberg construction" [15,16], introducing a third new coordinate $0\le u\le 1$. Accordingly all the quantities with *hats* are associated with the total extended three-dimensional manifold, and $\widehat G_{\mu\nu\rho}$ is a function of $\widehat X^\mu(\sigma,u)$, such that $\widehat X^\mu(\sigma,0)=0,\ \widehat X^\mu(\sigma,1)=X^\mu(\sigma)$ [15,16]. We have also identified these three-dimensional coordinates with the $D=(2,2)$ ones by $\sigma^1=\varphi,\ \sigma^2=\theta,\ u=\psi/(2\pi)$. Then the second expression of (3.22) contains nothing else than the Jacobian of this "coordinate transformation" from $(\sigma^1,\,\sigma^2,\,u)$ to $(\widehat X^\mu)$. Now the total contribution to the "phase-shift" in the string path-integral by our instanton between the "time" interval $1\le r<\infty$ is [16]

$$P(\infty)-P(1) = -6\pi \quad . \tag{3.23}$$

Since this is a multiple of 2π, our exact solutions are consistent with the $N=2$ superstring as its background! Needless to say, we have also used the above-mentioned replacement

[6] We have included the effect of YM gauge anomaly in the CS term by the GS mechanism [17], when $3\partial_{[\mu}B_{\nu\rho]}$ is converted into $\widehat G_{\mu\nu\rho}$ in (3.22). See Ref. [16] for the details.

(Wick-rotation) for the ϑ-integral. Interestingly, the integral in (3.22) is proportional to (3.20). This result has strong indication of topological significance and consistency of the $N=2$ string theory [4] formulated on our SDSG + SDSYM + SDTM background. Notice also that our computation involves various numerical factors, showing the powerful usage of our *canonical* notation.

The prescription we utilized above to get the $SL(2)$ SDYM from the SDG solution is universally applied to other cases, such as the Taub-Nut solution [18]. Some subtlety arises only when the Lorentz connection $\omega_\mu{}^{mn}$ is *not* self-dual, even though $R_{\mu\nu}{}^{mn}$ *is* self-dual. In such cases, we can always arrange $\omega_\mu{}^{mn}$ by appropriate Lorentz transformation such that it is manifestly self-dual [14]. For such self-dual ω's, the identification (3.4) is straightforward to get an exact solution for $SL(2)$ SDSYM.

4. *Concluding Remarks.* In this Letter we have presented a set of exact solutions for the coupled system of $N=1$ SDSG + SDSYM + SDTM on the EH gravitational instanton background for the first time. In our system, the dilaton field Φ played a peculiar role as a part of the SDTM, in particular by the special coupling to the SDSYM *via* the CS term in the third-rank field strength $G_{\mu\nu\rho}$. We also stress that our field equations for the SDTM is required by supersymmetry combined with the SD condition. Our *canonical* set of field equations are important for our derivations, due to the manifest SD for the Riemann tensor, which was obscure in other systems such as in Ref. [4].

In this Letter we put the fermionic backgrounds to be trivial, and eventually the background solutions have *no* manifest supersymmetry. However, we emphasize that our background exact solutions are *consistent* with supersymmetry, in the sense that they satisfy *supercovariant* field equations (2.1) - (2.6).

Even though our exact solutions are based on the $N=1$ SDSG + SDSYM + SDTM system, they will be important also in the PS-formulation [6,10] for extended supersymmetries with some additional multiplier fields, as well as in other formulations [6,12]. This is because the field equations for the non-multiplier fields stay exactly the same as our system after appropriate truncation into $N=1$ supersymmetry, being satisfied by the same exact solutions. For example, out of the 70 scalars in the $N=8$ SDSG multiplets [10], our SDTM emerges after appropriate duality-transformations.

In this Letter we have also checked the consistency of our exact solutions as a background for $N=2$ superstring theory, by inspecting the contribution to the WZW-term in the string path-integral. Remarkably our instanton solution contributes only $-6\pi i$ as a phase-shift after a Wick-rotation, indicating the validity of our solutions as consistent $N=2$ superstring background. This result reflects non-trivial *topological* aspects of the system, different from other *perturbative* features such as the β-functions treated in Ref. [4], providing an independent test for our background together with our GS σ-model [4] itself.

To our knowledge, our dilaton solution is the first peculiar example of its exact solutions, which is directly related to the antisymmetric tensor $B_{\mu\nu}$ in a non-trivial way, also as the consistent background for the $N = 2$ superstring.

As a by-product of our exact solutions themselves, we gave a general algorithm for getting an exact solution for SDYM for the gauge group $SL(2)$ out of any known SDG solution, as a modification of the method known for Euclidean space-times [13]. For example, we can repeat the same derivation for the Taub-Nut background [18] for the same field content.

Another useful application of our results is to the Euclidean *compact* space-time manifold [14], which has more advantages such as the topological indices or finite actions. After appropriate Wick-rotation, we can get the Euclidean version of our exact solutions.

It is well-known that the SDYM and SDG produce non-linear integrable field equations [2,7]. It is then natural to expect that their *supersymmetric* generalizations are also integrable.[7] As a matter of fact, $D = (2, 2)$ self-dual gauge theories can be identified with *two-dimensional* non-linear sigma-models on twistor surfaces with infinite dimensional algebra such as $\lim_{n\to\infty} SL(n)$ [8], and this connection should be supersymmetrized. The resulting gauge symmetries are the area-preserving diffeomorphisms, indicating possible link to the W_∞-algebra [9]. Our exact solutions are the first explicit examples for the SDSG + SDSYM + SDTM, which can connect these self-dual gauge theories.

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[7]Actually in our recent paper [19] we have shown that $N = 1$ SDSYM theory will embed $N = 1$ and $N = 2$ supersymmetric KdV equations and $N = 1$ supersymmetric Toda theory in two-dimensions, after appropriate dimensional reductions.

References

[1] H. Ooguri and C. Vafa, Mod. Phys. Lett. **A5** (1990) 1389; Nucl. Phys. **B361** (1991) 469; *ibid.* **367** (1991) 83;
H. Nishino and S.J. Gates, Jr., Mod. Phys. Lett. **A7** (1992) 2543.

[2] A.A. Belavin, A. M. Polyakov, A. Schwartz and Y. Tyupkin, Phys. Lett. **59B** (1975) 85;
R.S. Ward, Phys. Lett. **61B** (1977) 81;
M.F. Atiyah and R.S. Ward, Comm. Math. Phys. **55** (1977) 117;
E.F. Corrigan, D.B. Fairlie, R.C. Yates and P. Goddard, Comm. Math. Phys. **58** (1978) 223;
E. Witten, Phys. Rev. Lett. **38** (1977) 121;
A.N. Leznov and M.V. Saveliev, Comm. Math. Phys. **74** (1980) 111;
L. Mason and G. Sparling, Phys. Lett. **137B** (1989) 29;
I. Bakas and D.A. Depireux, Mod. Phys. Lett. **A6** (1991) 399; *ibid.* 1561; 2351.

[3] S.V. Ketov, S.J. Gates and H. Nishino, Maryland preprint, UMDEPP 92–163 (February 1992).

[4] H. Nishino, S. J. Gates, Jr. and S. V. Ketov, Maryland preprint, UMDEPP 92–171 (February 1992).

[5] S. J. Gates, Jr., H. Nishino and S. V. Ketov, Maryland preprint, UMDEPP 92–187 (March 1992), to appear in Phys. Lett. B.

[6] S.J. Ketov, H. Nishino and S.J. Gates, Jr., Maryland preprint, UMDEPP 92–211 (June 1992), to appear in Nucl. Phys. B.

[7] M. F. Atiyah, unpublished;
R. S. Ward, Phil. Trans. Roy. Lond. **A315** (1985) 451;
N. J. Hitchin, Proc. Lond. Math. Soc. **55** (1987) 59.

[8] Q.-H. Park, Phys. Lett. **238B** (1990) 287; *ibid.* **257B** (1991) 105;
Cambridge preprint, DAMTP R-91/12 (October, 1991).

[9] See, e.g., E. Sezgin, Texas A & M preprint, CTP-TAMU-13/92 (February, 1991).

[10] W. Siegel, Stony Brook preprint, ITP-SB-92-31 (July 1992).

[11] T. Eguchi and A. Hanson, Ann. of Phys. **120** (1979) 82;
E. Calabi, Ann. Sci. Ec. Norm. Sup. **12** (1979) 269.

[12] H. Nishino, Maryland preprint, in preparation (October 1992).

[13] J.M. Charap and M. Duff, Phys. Lett. **69B** (1977) 445.

[14] G.W. Gibbons and S.W. Hawking, Comm. Math. Phys. **66** (1979) 291;
T. Eguchi, P.B. Gilkey and A.J. Hanson, Phys. Rep. **6C** (1980) 213.

[15] S.J. Gates, Jr. and H. Nishino, Phys. Lett. **173B** (1986) 46 and 52.

[16] R. Rohm and E. Witten, Ann. of Phys. **170** (1986) 454.

[17] M. Green and J.H. Schwarz, Phys. Lett. **149B** (1984) 117.

[18] S.W. Hawking, Phys. Lett. **60A** (1977) 81.

[19] H. Nishino and S.J. Gates, Jr., Maryland preprint, UMDEPP 93–51 (September, 1992).